A sharp estimate a la Calderon-Zygmund for the p-Laplacian

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A SHARP ESTIMATE À LA CALDERÓN-ZYGMUND FOR THE $p-$LAPLACIAN

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ABSTRACT. We consider local weak solutions of the Poisson equation for the $p-$Laplace operator. We prove a higher differentiability result, under an essentially sharp condition on the right-hand side. The result comes with a local scaling invariant a priori estimate.

CONTENTS

1. INTRODUCTION

1.1. **The problem.** In this paper we are concerned with local or global $W^{1,p}$ solutions to the Poisson equation for the $p-$Laplace operator, i.e.

$$-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\,\nabla u) = f, \qquad \text{in } \Omega, \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$ open set. Our analysis is confined to the super-quadratic case, i.e. throughout the whole paper we consider $p > 2$. For $f \equiv 0$, we know that

$$|\nabla u|^{\frac{p-2}{2}}\,\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N), \tag{1.2}$$

This is a well-known regularity result which dates back to Uhlenbeck, see [13, Lemma 3.1]. We refer to [6, Proposition 3.1] for a generalization of this result. If f is smooth enough, the same

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result is easily seen to be still true. For example, it is sufficient to take

$$f \in W^{1,p'}_{\rm loc}(\Omega), \tag{1.3}$$

where $p' = p/(p-1)$. However, it is easy to guess that assumption (1.3) is far from being optimal: in the limit case $p = 2$, (1.2) boils down to

$$\nabla u \in W^{1,2}_{\rm loc}(\Omega; \mathbb{R}^N).$$

Then from *Calderón-Zygmund estimates* for the Laplacian, we know that this is true if (and only if)

$$f \in L^2_{\rm loc}(\Omega).$$

Thus in this case $f \in L^2_{\rm loc}$ would be the sharp assumption in order to get Uhlenbeck's result. The main concern of this work is to prove Uhlenbeck's result for solutions of (1.1), under sharp assumptions on f.

1.2. **The main result.** In this paper we prove the following result. We refer to Section 2 for the notation.

Theorem 1.1. *Let $p > 2$ and let $U \in W^{1,p}_{\rm loc}(\Omega)$ be a local weak solution of equation (1.1). If*

$$f \in W^{s,p'}_{\rm loc}(\Omega) \qquad \text{with } \frac{p-2}{p} < s \leq 1,$$

then

$$\mathcal{V} := |\nabla U|^{\frac{p-2}{2}}\, \nabla U \in W^{1,2}_{\rm loc}(\Omega; \mathbb{R}^N),$$

and

$$\nabla U \in W^{\sigma,p}_{\rm loc}(\Omega; \mathbb{R}^N), \qquad \text{for } 0 < \sigma < \frac{2}{p}.$$

Moreover, for every pair of concentric balls $B_r \Subset B_R \Subset \Omega$ and every $j = 1, \dots, N$ we have the local scaling invariant estimate

$$\int_{B_r} \left|\mathcal{V}_{x_j}\right|^2 dx \leq \frac{C}{(R-r)^2} \int_{B_R} |\nabla U|^p\, dx + C \left(R^{\left(s-\frac{p-2}{p}\right)}\, [f]_{W^{s,p'}(B_R)} \right)^{p'}, \tag{1.4}$$

for a constant $C = C(N, p, s) > 0$ which blows-up as $s \searrow (p-2)/p$.

Remark 1.2 (Sharpness of the assumption)**.** The assumption on f in the previous result is essentially sharp, in the sense that *the result is false for $s < (p-2)/p$*, see Section 5 for an example. Also observe that

$$\frac{p-2}{p} \searrow 0 \quad \text{as} \quad p \searrow 2,$$

thus the assumption on f is consistent with the case of the Laplacian recalled above.

Remark 1.3 (Comparison with other results)**.** The interplay between regularity of the right-hand side f and that of the vector field $\mathcal{V}$ has been considered in detail by Mingione in [5]. However, our Theorem 1.1 does not superpose with the results of [5]. Indeed, the point of view in [5] is slightly different: the main concern there is to obtain (fractional) differentiability of the vector field

$$\mathcal{V} := |\nabla U|^{\frac{p-2}{2}}\, \nabla U,$$

when f is not regular. In particular, in [5] the right-hand side f may not belong to the relevant dual Sobolev space and the concept of solution to (1.1) has to be carefully defined.

In order to give a flavour of the results in [5], we recall that [5, Theorem 1.3] proves that if $2 < p \le N$ and f is a Radon measure, then

$$\mathcal{V} \in W^{\frac{\tau}{2},\frac{2}{p'}}_{\rm loc}(\Omega;\mathbb{R}^N), \qquad \text{for every } 0<\tau<p'.$$

When the Radon measure f is more regular, accordingly one can improve the differentibility of $\mathcal{V}$. For example, [5, Theorem 1.6] proves that if $2 < p \le N$ and $f \in L^{1,\lambda}(\Omega)$ with $N-p < \lambda \le N$, then[1]

$$\mathcal{V} \in W^{\frac{\tau}{2},2}_{\rm loc}(\Omega;\mathbb{R}^N), \qquad \text{for every } 0<\tau<p'\left(1-\frac{N-\lambda}{p}\right).$$

Here $L^{1,\lambda}(\Omega)$ is the usual *Morrey space*. In this case, the assumption on f guarantees that a solution to (1.1) can be defined in variational sense, we refer to [5] for more details.

Some prior results are also due to J. Simon, who proved for example the global regularity

$$\mathcal{V} \in W^{\frac{\tau}{2},2}(\mathbb{R}^N), \qquad \text{for every } 0<\tau<p',$$

for solutions U in the whole $\mathbb{R}^N$ with right-hand side $f \in L^{p'}(\mathbb{R}^N)$, see [11, Theorem 8.1]. Finally, even if it is concerned with the solution U rather than the vector field $\mathcal{V}$, we wish to mention a result by Savaré contained in [10]. This paper is concerned with *global regularity* for solutions of (1.1) satisfying homogeneous Dirichlet boundary conditions. In [10, Theorem 2], it is shown that

$$f \in W^{-1+\frac{\tau}{p'},p'}(\Omega) \quad \Longrightarrow \quad U \in W^{1+\frac{\tau}{p},p}(\Omega), \qquad \text{for every } 0<\tau<1,$$

when $\partial\Omega$ is Lipschitz continuous. This gives a regularity gain on the solution U of

$$2+\tau\left(\frac{1}{p}-\frac{1}{p'}\right),$$

orders of differentiability, compared to the right-hand side f. It is interesting to notice that by formally taking $\tau = 2$ in the previous implication, this essentially gives the regularity gain of Theorem 1.1.

1.3. **About the proof.** Let us try to explain in a nutshell the key point of estimate (1.4). For the sake of simplicity, let us assume that U is smooth (i.e. $U \in C^2$) and explain how to arrive at the a priori estimate (1.4). The rigourous proof is then based on a standard approximation procedure.

For ease of notation, we set

$$G(z) = \frac{|z|^p}{p}, \qquad \text{for } z \in \mathbb{R}^N,$$

then $U \in W^{1,p}_{\rm loc}(\Omega)$ verifies

$$\int \langle \nabla G(\nabla U), \nabla \varphi\rangle\, dx = \int f\,\varphi,$$

for every compactly supported test function φ. Uhlenbeck's result is just based on differentiating this equation in direction x_j and then testing it against[2] U_{x_j}. This yields

$$\int \langle D^2 G(\nabla U)\,\nabla U_{x_j}, \nabla U_{x_j}\rangle\, dx = \int f_{x_j}\, U_{x_j}\, dx. \tag{1.5}$$

[1]The result of [5, Theorem 1.6] is not stated for $\mathcal{V}$, but rather directly for ∇U. However, an inspection of the proof reveals that one has the claimed regularity of $\mathcal{V}$, see [5, proof of Theorem 1.6, page 33].

[2]Of course, this test function is not compactly supported. Actually, to make it admissible we have to multiply it by a cut-off function, see Proposition 3.2. This introduces some lower-order terms in the estimate, which are not essential at this level and would just hide the idea of the proof.

By using the convexity properties of G, it is easy to see that

$$\int |\mathcal{V}_{x_j}|^2\,dx = \int \left|\left(|\nabla U|^{\frac{p-2}{2}}\,\nabla U\right)_{x_j}\right|^2 dx \lesssim \int \langle D^2G(\nabla U)\,\nabla U_{x_j}, \nabla U_{x_j}\rangle\,dx.$$

The main difficulty is now to estimate the right-hand side of (1.5), without using first order derivatives of f. A first naïve idea would be to integrate by parts: of course, this can not work, since this would let appear the Hessian of U on which we do not have any estimate. A more clever strategy is to *integrate by parts in fractional sense*, i.e. use a duality-based inequality of the form

$$\left|\int f_{x_j}\,U_{x_j}\,dx\right| \le \|f_{x_j}\|_{W^{s-1,p'}}\,\|U_{x_j}\|_{W^{1-s,p}},$$

where $W^{s-1,p'}$ is just the topological dual of $W^{1-s,p}$. The main point is then to prove that

"*taking a fractional derivative of negative order of* f_{x_j}

gives a fractional derivative of positive order"

i.e. we use that

$$\|f_{x_j}\|_{W^{s-1,p'}} \lesssim \|f\|_{W^{s,p'}}, \tag{1.6}$$

see Theorem 2.6 below.

In order to conclude, we still have to control the term containing fractional derivatives of U_{x_j}. This can be absorbed in the left-hand side, once we notice that U_{x_j} is the composition of $\mathcal{V}$ of with a Hölder function. More precisely, we have

$$U_{x_j} \simeq |\mathcal{V}_j|^{\frac{2}{p}},$$

thus if $1-s<2/p$ we get

$$\begin{aligned}\|U_{x_j}\|^p_{W^{1-s,p}} &\lesssim \sup_{|h|>0}\int \frac{|U_{x_j}(\cdot+h)-U_{x_j}|^p}{|h|^2}\,dx \\ &\lesssim \sup_{|h|>0}\int \frac{|\mathcal{V}_j(\cdot+h)-\mathcal{V}_j|^2}{|h|^2}\,dx \simeq \int |\nabla\mathcal{V}_j|^2\,dx \lesssim \int |\mathcal{V}_{x_j}|^2\,dx.\end{aligned}$$

This would permit to obtain the desired estimate on $\mathcal{V}$, under the standing assumption on f.

Remark 1.4. Actually, the genesis of Theorem 1.1 is somehow different from the above sketched proof. Indeed, the fact that fractional Sobolev regularity of f should be enough to obtain $\mathcal{V}\in W^{1,2}$ appeared as natural in the framework of the *regularity via duality* strategy presented in [9]. This is a general strategy, first used in the much harder context of variational methods for the incompressible Euler equation by Brenier in [3], and then re-applied to Mean-Field Games (see for instance [8] for a simple case where this strategy is easy to understand). This strategy allows to prove estimates on the incremental ratios

$$\frac{u(\cdot+h)-u}{h},$$

of the solutions u of convex variational problems by using the non-optimal function $u(\cdot+h)$ in the corresponding primal-dual optimality conditions.

In our main result above we make the restriction $p>2$. For completeness, let us comment on the sub-quadratic case.

Remark 1.5 (The case $1 < p < 2$)**.** The sub-quadratic case is simpler, indeed we already know that in this case

$$f \in L^{p'}_{\rm loc}(\Omega) \quad \Longrightarrow \quad \mathcal{V} \in W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^N) \quad \Longrightarrow \quad \nabla U \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^N),$$

see for example[3] [4, Theorem] by de Thelin. We can have an idea of the proof of this result by still following the guidelines sketched above. We treat the left-hand side of (1.5) as before, while on the right-hand side one now performs an integration by parts and uses Hölder's inequality. These yield

$$\left|\int f_{x_j}\, U_{x_j}\, dx\right| \le \|f\|_{L^{p'}}\, \|U_{x_j\,x_j}\|_{L^p} \qquad \text{and} \qquad \|U_{x_j\,x_j}\|_{L^p} \lesssim \|\nabla \mathcal{V}\|_{L^2}\, \|\nabla U\|_{L^p}^{\frac{2-p}{2}},$$

and the term containing the gradient of $\nabla\mathcal{V}$ can be absorbed in the left-hand side. Observe that

$$p' \searrow 2 \qquad \text{as} \qquad p \nearrow 2,$$

thus again the assumption on f is consistent with the case of the Laplacian.

1.4. **Plan of the paper.** We set the notation and recall the basic facts on functional spaces in Section 2. Here, the important point is Theorem 2.6, which proves inequality (1.6). In Section 3 we consider a regularization of equation (1.1) and prove a Sobolev estimate, independent of the regularization parameter (Proposition 3.2). Then in Section 4 we show how to take the estimate to the limit and achieve the proof of Theorem 1.1. We show with an example that our assumption is essentially sharp: this is Section 5. The paper closes with an appendix containing some technical tools needed for the proof of Theorem 2.6.

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2. Preliminaries

2.1. **Notation.** For a measurable function $\psi : \mathbb{R}^N \to \mathbb{R}^k$ and a vector $h \in \mathbb{R}^N$, we define

$$\psi_h(x) = \psi(x+h), \qquad \delta_h\psi(x) = \psi_h(x) - \psi(x),$$

and

$$\delta^2_h\psi(x) = \delta_h(\delta_h\psi(x)) = \psi_{2h}(x) + \psi(x) - 2\,\psi_h(x).$$

We consider the two vector-valued functions

$$\nabla G(z) = |z|^{p-2}\, z \qquad \text{and} \qquad V(z) = |z|^{\frac{p-2}{2}}\, z, \qquad \text{for } z \in \mathbb{R}^N.$$

The following inequalities are well-known, we omit the proof.

Lemma 2.1. *Let $p > 2$, for every $z, w \in \mathbb{R}^N$ we have*

$$|z - w| \le C_1\, |V(z) - V(w)|^{\frac{2}{p}}, \tag{2.1}$$

$$\Big|V(z) - V(w)\Big| \le C_2\, \left(|z|^{\frac{p-2}{2}} + |w|^{\frac{p-2}{2}}\right) |z - w|, \tag{2.2}$$

for some $C_1 = C_1(p) > 0$ and $C_2 = C_2(p) > 0$.

[3]In [4] as well the result is stated directly for ∇U. However, it is easily seen that the very same proof leads to the stronger result for $\mathcal{V}$.

2.2. **Functional spaces.** We recall the definition of some fractional Sobolev spaces needed in the sequel. Let $1 \le q < \infty$ and let $\psi \in L^q(\mathbb{R}^N)$, for $0 < \beta \le 1$ we set

$$[\psi]_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)} := \sup_{|h|>0} \left\| \frac{\delta_h \psi}{|h|^\beta} \right\|_{L^q(\mathbb{R}^N)},$$

and for $0 < \beta < 2$

$$[\psi]_{\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)} := \sup_{|h|>0} \left\| \frac{\delta^2_h \psi}{|h|^\beta} \right\|_{L^q(\mathbb{R}^N)}.$$

We then introduce the two Besov-type spaces

$$\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N) = \left\{ \psi \in L^q(\mathbb{R}^N) \, : \, [\psi]_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)} < +\infty \right\}, \qquad 0 < \beta \le 1,$$

and

$$\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N) = \left\{ \psi \in L^q(\mathbb{R}^N) \, : \, [\psi]_{\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)} < +\infty \right\}, \qquad 0 < \beta < 2.$$

We also need the *Sobolev-Slobodeckiĭ space*

$$W^{\beta,q}(\mathbb{R}^N) = \left\{ \psi \in L^q(\mathbb{R}^N) \, : \, [\psi]_{W^{\beta,q}(\mathbb{R}^N)} < +\infty \right\}, \qquad 0 < \beta < 1,$$

where the seminorm $[\,\cdot\,]_{W^{\beta,q}(\mathbb{R}^N)}$ is defined by

$$[\psi]_{W^{\beta,q}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x) - \psi(y)|^q}{|x - y|^{N+\beta\, q}} \, dx \, dy \right)^{\frac{1}{q}}.$$

For $1 < \beta < 2$, the space $W^{\beta,q}(\mathbb{R}^N)$ consists of

$$W^{\beta,q}(\mathbb{R}^N) = \left\{ \psi \in L^q(\mathbb{R}^N) \, : \, \nabla \psi \in W^{\beta-1,q}(\mathbb{R}^N) \right\}.$$

More generally, if $\Omega \subset \mathbb{R}^N$ is an open set, the space $W^{\beta,q}(\Omega)$ is defined by

$$W^{\beta,q}(\Omega) = \left\{ \psi \in L^q(\Omega) \, : \, [\psi]_{W^{\beta,q}(\Omega)} < +\infty \right\}, \qquad 0 < \beta < 1,$$

and the seminorm $[\,\cdot\,]_{W^{\beta,q}(\Omega)}$ is defined accordingly. We endow this space with the norm

$$\|\psi\|_{W^{\beta,q}(\Omega)} = \|\psi\|_{L^q(\Omega)} + [\psi]_{W^{\beta,q}(\Omega)}.$$

For an open bounded set $\Omega \subset \mathbb{R}^N$ and $0 < \beta \le 1$, the *homogeneous Sobolev-Slobodeckiĭ space* $\mathcal{D}^{\beta,q}_0(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\psi \mapsto \|\psi\|_{\mathcal{D}^{\beta,q}_0(\Omega)} := [\psi]_{W^{\beta,q}(\mathbb{R}^N)} \qquad \text{if } 0 < \beta < 1,$$

or

$$\psi \mapsto \|\psi\|_{\mathcal{D}^{1,q}_0(\Omega)} := \|\nabla \psi\|_{L^q(\mathbb{R}^N)} \qquad \text{if } \beta = 1.$$

Finally, for $0 < \beta \le 1$ the topological dual of $\mathcal{D}^{\beta,q}_0(\Omega)$ will be denoted by $\mathcal{D}^{-\beta,q'}(\Omega)$. We endow this space with the natural dual norm, defined by

$$\|F\|_{W^{-\beta,q'}(\Omega)} = \sup \left\{ |\langle F, \varphi \rangle| \, : \, \varphi \in C^\infty_0(\Omega) \ \text{ with } \ \|\psi\|_{\mathcal{D}^{\beta,q}_0(\Omega)} \le 1 \right\}, \qquad F \in \mathcal{D}^{-\beta,q'}(\Omega).$$

Remark 2.2. It is not difficult to show that for $0<\beta\le 1$ we have

$$\mathcal{D}_0^{\beta,q}(\Omega)\hookrightarrow L^q(\Omega),$$

thanks to Poincaré inequality. The latter reads as

$$\|u\|_{L^q(\Omega)}^q\le C\,|\Omega|^{\frac{\beta\,q}{N}}\,\|u\|^q_{\mathcal{D}_0^{\beta,q}(\Omega)},$$

with $C=C(N,\beta,q)>0$.

2.3. **Embedding results.** In order to make the paper self-contained, we recall some functional inequalities needed in dealing with fractional Sobolev spaces.

Lemma 2.3. *Let $0<\beta<1$ and $1\le q<\infty$, then we have the continuous embedding*

$$\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)\hookrightarrow\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N).$$

More precisley, for every $\psi\in\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)$ we have

$$[\psi]_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)}\le\frac{C}{1-\beta}\,[\psi]_{\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)},$$

for some constant $C=C(N,q)>0$.

Proof. We already know that

$$[\psi]_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)}\le\frac{C}{1-\beta}\,\left[[\psi]_{\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)}+\|\psi\|_{L^q(\mathbb{R}^N)}\right],$$

se for example [2, Lemma 2.3]. In particular, by using this inequality for $\psi^\lambda(x)=\psi(\lambda\,x)$ with $\lambda>0$, after a change of variable we obtain

$$\lambda^{\beta-\frac{N}{q}}\,[\psi]_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)}\le\frac{C}{1-\beta}\,\left[\lambda^{\beta-\frac{N}{q}}\,[\psi]_{\mathcal{B}^{\beta,q}_\infty(\mathbb{R}^N)}+\lambda^{-\frac{N}{q}}\,\|\psi\|_{L^q(\mathbb{R}^N)}\right].$$

By multiplying by $\lambda^{N/q-\beta}$ and taking the limit as λ goes to $+\infty$, we get the desired inequality. $\square$

Proposition 2.4. *Let $1\le q<\infty$ and $0<\alpha<\beta<1$. We have the continuous embedding*

$$\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)\hookrightarrow W^{\alpha,q}(\mathbb{R}^N).$$

More precisely, for every $\psi\in\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)$ we have

$$[\psi]^q_{W^{\alpha,q}(\mathbb{R}^N)}\le C\,\frac{\beta}{(\beta-\alpha)\,\alpha}\,\left([\psi]^q_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)}\right)^{\frac{\alpha}{\beta}}\,\left(\|\psi\|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{\beta-\alpha}{\beta}},$$

for some constant $C=C(N,q)>0$.

Proof. Let us fix $h_0>0$, by appealing for example to [2, Proposition 2.7], we already know that

$$[\psi]^q_{W^{\alpha,q}(\mathbb{R}^N)}\le C\,\left(\frac{h_0^{(\beta-\alpha)\,q}}{\beta-\alpha}\,\sup_{0<|h|<h_0}\left\|\frac{\delta_h\psi}{|h|^\beta}\right\|^q_{L^q(\mathbb{R}^N)}+\frac{h_0^{-\alpha\,q}}{\alpha}\,\|\psi\|^q_{L^q(\mathbb{R}^N)}\right).$$

for a constant C depending on N and q only. If we now optimize the right-hand side in h_0 we get the desired conclusion. $\square$

Proposition 2.5. *Let $1 \le q < \infty$ and $0 < \alpha < \beta < 1$. We have the continuous embedding*

$$\mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N) \hookrightarrow W^{1+\alpha,q}(\mathbb{R}^N).$$

In particular, for every $\psi \in \mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N)$ we have $\nabla\psi \in W^{\alpha,q}(\mathbb{R}^N)$, with the following estimates

$$\|\nabla\psi\|^q_{L^q(\mathbb{R}^N)} \le \frac{C}{\beta^{\frac{\beta+q}{\beta+1}}}\,\left(\|\psi\|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{\beta}{\beta+1}}\,\left([\psi]^q_{\mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N)}\right)^{\frac{1}{\beta+1}}, \tag{2.3}$$

for some $C = C(N,q) > 0$, and

$$[\nabla\psi]^q_{W^{\alpha,q}(\mathbb{R}^N)} \le C\left([\psi]^q_{\mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N)}\right)^{\frac{\alpha+1}{\beta+1}}\,\left(\|\psi\|^q_{L^q(\mathbb{R}^N)}\right)^{\frac{\beta-\alpha}{\beta+1}}, \tag{2.4}$$

for some $C = C(N,q,\alpha,\beta) > 0$, which blows up as $\alpha \nearrow \beta$, $\beta \searrow 0$ or $\beta \nearrow 1$.

Proof. We already know that

$$\|\nabla\psi\|^q_{L^q(\mathbb{R}^N)} \le C\,\|\psi\|^q_{L^q(\mathbb{R}^N)} + \frac{C}{\beta^q}\,[\psi]^q_{\mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N)},$$

see for example [2, Proposition 2.4]. By replacing ψ with the rescaled function $\psi^\lambda(x) = \psi(\lambda\,x)$ and optimizing in λ, we get (2.3).

For the second estimate (2.4), it is sufficient to observe that

$$[\nabla\psi]^q_{\mathcal{N}^{\beta,q}_\infty(\mathbb{R}^N)} \le \frac{C}{\beta^q\,(1-\beta)^q}\,[\psi]^q_{\mathcal{B}^{1+\beta,q}_\infty(\mathbb{R}^N)},$$

again by [2, Proposition 2.4]. Then by using Proposition 2.4 for $\nabla\psi$ and (2.3), we get the desired conclusion. □

2.4. **An inequality for negative norms.** As explained in the Introduction, a crucial rôle in the proof of Theorem 1.1 is played by the following weak generalization of the so-called *Nečas' negative norm Theorem* (which corresponds to $\beta = 0$, see [7]).

Theorem 2.6. *Let $0 < \beta < 1$ and $1 < q < \infty$. Let $B \subset \mathbb{R}^N$ be an open ball, for every $f \in W^{\beta,q}(B)$ we have*

$$\|f_{x_j}\|_{\mathcal{D}^{\beta-1,q}(B)} \le C\,[f]_{W^{\beta,q}(B)}, \qquad j = 1,\dots,N, \tag{2.5}$$

for a constant $C = C(N,\beta,q) > 0$.

Proof. We explain the guidelines of the proof, by referring the reader to Appendix A for the missing details. We first prove

$$\|f_{x_j}\|_{\mathcal{D}^{\beta-1,q}(B)} \le C\left(\|f\|_{L^q(B)} + [f]_{W^{\beta,q}(B)}\right), \qquad j = 1,\dots,N. \tag{2.6}$$

This estimate says that the linear operator

$$T_j : W^{\beta,q}(B) \to \mathcal{D}^{\beta-1,q}(B),$$

defined by the weak $j-$th derivative is continuous. It is easy to see that T_j is continuous as an operator defined on $W^{1,q}(B)$ and $L^q(B)$. More precisely, the following operators

$$\begin{aligned} T_j :\ & W^{1,q}(B) \ \to\ L^q(B) \\ T_j :\ & L^q(B) \ \to\ \mathcal{D}^{-1,q}(B) \end{aligned}$$

are linear and continuous. In other words, inequality (2.6) is true for the extremal cases $\beta = 0$ and $\beta = 1$. We observe that $W^{\beta,q}(B)$ is an interpolation space between $W^{1,q}(B)$ and $L^q(B)$, i.e.

$$W^{\beta,q}(B) = \big(L^q(B), W^{1,q}(B)\big)_{\beta,q},$$

see Definition A.2 for the notation. We can then obtain quite easily that T_j is continuous from $W^{\beta,q}(B)$ to the interpolation space between $\mathcal{D}^{-1,q}(B)$ and $L^q(B)$, i.e.

$$\big(\mathcal{D}^{-1,q}(B), L^q(B)\big)_{\beta,q},$$

see Lemma A.5. Such a space can be computed explicitely: as one may expect, it coincides with the dual Sobolev-Slobodeckiĭ space $\mathcal{D}^{\beta-1,q}(B)$ (see Lemma A.6). This proves inequality (2.6).

In order to get (2.5) and conclude, it is now sufficient to use a standard scaling argument. Let us assume for simplicity that B is centered at the origin, for $f \in W^{\beta,q}(B)$ and every $\lambda > 0$, we define $f_\lambda(x) = f(x/\lambda)$. This belongs to $W^{\beta,q}(\lambda\, B)$, then from (2.6) and the scaling properties of the norms, we get

$$\begin{aligned}\lambda^{\frac{N}{q}-\beta}\, \|f_{x_j}\|_{\mathcal{D}^{\beta-1,q}(B)} = \|(f_\lambda)_{x_j}\|_{\mathcal{D}^{\beta-1,q}(\lambda\, B)} &\le C\, \Big(\|f_\lambda\|_{L^q(\lambda\, B)} + [f_\lambda]_{W^{\beta,q}(\lambda\, B)}\Big)\\ &= C\, \Big(\lambda^{\frac{N}{q}}\, \|f\|_{L^q(B)} + \lambda^{\frac{N}{q}-\beta}\, [f]_{W^{\beta,q}(B)}\Big).\end{aligned}$$

If we multiply by $\lambda^{\beta-N/q}$ and then let λ go to 0, we finally get the desired estimate. $\square$

3. Estimates for a regularized problem

Let U and f be as in the statement of Theorem 1.1. Let $B \Subset \Omega$ be an open ball, for every $\varepsilon > 0$ we consider the problem

$$\left\{\begin{array}{rcll} -\mathrm{div} \nabla G_\varepsilon(\nabla u) & = & f_\varepsilon, & \text{in } B,\\ u & = & U, & \text{on } \partial B,\end{array}\right. \tag{3.1}$$

where:

- $G_\varepsilon(z) = \dfrac{1}{p}\,(\varepsilon + |z|^2)^{\frac{p}{2}}$, for every $z \in \mathbb{R}^N$;
- $f_\varepsilon = f * \varrho_\varepsilon$ and $\{\varrho_\varepsilon\}_{\varepsilon>0}$ is a family of standard compactly supported C^∞ mollifiers.

Problem (3.1) admits a unique solution $u_\varepsilon \in W^{1,p}(B)$, which is locally smooth in B by standard elliptic regularity.

Proposition 3.1 (Uniform energy estimate)**.** *With the notation above, we have*

$$\int_B |\nabla u_\varepsilon|^p\, dx \le C\, \varepsilon^{\frac{p-1}{2}} \int_B |\nabla U|\, dx + C \int_B |\nabla U|^p\, dx + C\, |B|^{\frac{p'}{N}}\, \|f_\varepsilon\|^{p'}_{L^{p'}(B)}, \tag{3.2}$$

for some $C = C(N,p) > 0$.

Proof. The proof is standard, we include it for completeness. We take the weak formulation of (3.1)

$$\int \langle \nabla G_\varepsilon(\nabla u_\varepsilon), \nabla\varphi\rangle\, dx = \int f_\varepsilon\, \varphi\, dx, \qquad \text{for every } \varphi \in W^{1,p}_0(B),$$

and insert the test function $\varphi = u_\varepsilon - U$. This gives

$$\begin{aligned}\int_B \langle \nabla G_\varepsilon(\nabla u_\varepsilon), \nabla u_\varepsilon\rangle\, dx &= \int_B \langle \nabla G_\varepsilon(\nabla u_\varepsilon), \nabla U\rangle\, dx + \int_B f_\varepsilon\,(u_\varepsilon - U)\, dx\\ &\le \int |\nabla G_\varepsilon(\nabla u_\varepsilon)|\, |\nabla U|\, dx + \|f_\varepsilon\|_{L^{p'}(B)}\, \|u_\varepsilon - U\|_{L^p(B)}.\end{aligned}$$

We then observe that

$$\langle \nabla G_\varepsilon(z), z\rangle \ge |z|^p, \qquad z\in\mathbb{R}^N,$$

and

$$|\nabla G_\varepsilon(z)| \le C\,\left(\varepsilon^{\frac{p-2}{2}}\,|z| + |z|^{p-1}\right) \le C\,\varepsilon^{\frac{p-1}{2}} + 2\,C\,|z|^{p-1}, \qquad z\in\mathbb{R}^N,$$

for some $C = C(p) > 0$. By using these inequalities, we get

$$\begin{aligned}\int_B |\nabla u_\varepsilon|^p\, dx &\le C\,\varepsilon^{\frac{p-1}{2}} \int_B |\nabla U|\, dx + C\int_B |\nabla u_\varepsilon|^{p-1}\, |\nabla U|\, dx + \|f_\varepsilon\|_{L^{p'}(B)}\, \|u_\varepsilon - U\|_{L^p(B)}\\ &\le C\,\varepsilon^{\frac{p-1}{2}} \int_B |\nabla U|\, dx + C\,\tau \int_B |\nabla u_\varepsilon|^p\, dx\\ &+ C\,\tau^{-\frac{1}{p-1}} \int_B |\nabla U|^p\, dx + C\,|B|^{\frac{1}{N}}\, \|f_\varepsilon\|_{L^{p'}(B)}\, \|\nabla u_\varepsilon - \nabla U\|_{L^p(B)},\end{aligned}$$

where we used Young's inequality in the second term and Poincaré's inequality in the last one. By taking $\tau > 0$ sufficiently small, we can then obtain

$$\begin{aligned}\int_B |\nabla u_\varepsilon|^p\, dx &\le C\,\varepsilon^{\frac{p-1}{2}} \int_B |\nabla U|\, dx + C \int_B |\nabla U|^p\, dx\\ &+ C\,|B|^{\frac{1}{N}}\, \|f_\varepsilon\|_{L^{p'}(B)}\, \|\nabla u_\varepsilon - \nabla U\|_{L^p(B)}.\end{aligned}$$

We can now use the triangle inequality on the last term and conclude with a further application of Young's inequality, in order to absorb the term containing ∇u_ε. We leave the details to the reader. □

The proof of Theorem 1.1 is crucially based on the following

Proposition 3.2 (Uniform Sobolev estimate)**.** *We set*

$$\mathcal{V}_\varepsilon := V(\nabla u_\varepsilon) = |\nabla u_\varepsilon|^{\frac{p-2}{2}}\, \nabla u_\varepsilon.$$

Let $(p-2)/p < s \le 1$, for every pair of concentric balls $B_r \Subset B_R \Subset B$ and every $j = 1,\dots,N$ we have

$$\int_{B_r} \left|(\mathcal{V}_\varepsilon)_{x_j}\right|^2 dx \le \frac{C}{(R-r)^2} \int_{B_R} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p}{2}}\, dx + C\left(R^{\left(s-\frac{p-2}{p}\right)}\, [f_\varepsilon]_{W^{s,p'}(B_R)}\right)^{p'}, \tag{3.3}$$

for a constant $C = C(N,p,s) > 0$ which blows up as $s \searrow (p-2)/p$.

Proof. In what follows, for notational simplicity we omit to indicate the dependence on $\varepsilon > 0$ and simply write

$$G, \quad u, \quad \mathcal{V} \quad \text{and} \quad f.$$

For $j \in \{1,\dots,N\}$, in the weak formulation of (3.1) we insert a test function of the form φ_{x_j} for $\varphi \in C^\infty_0(B)$. After an integration by parts, we obtain

$$\int \langle D^2 G(\nabla u)\, \nabla u_{x_j}, \nabla\varphi\rangle\, dx = \int f_{x_j}\, \varphi\, dx.$$

By density, this relation remains true for $\varphi \in W^{1,p}$ with compact support in B. We then insert the test function

$$\varphi = \zeta^2\, u_{x_j},$$

with $\zeta \in C_0^\infty(B_R)$ a standard cut-off function such that

$$0 \le \zeta \le 1, \qquad \zeta \equiv 1 \ \text{ in } B_r, \qquad |\nabla \zeta| \le \frac{C}{R-r}. \tag{3.4}$$

Thus we obtain

$$\begin{aligned}\int \langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j}\rangle\, \zeta^2\, dx &= -2 \int \langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla \zeta\rangle\, u_{x_j}\, \zeta\, dx \\ &\quad + \int f_{x_j}\, u_{x_j}\, \zeta^2\, dx.\end{aligned} \tag{3.5}$$

We first observe that the left-hand side is positive, since G is convex. As for the right-hand side, we have

$$\begin{aligned}\int \langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla \zeta\rangle\, u_{x_j}\, \zeta\, dx &\le \int \left|\langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla \zeta\rangle\right|\, |u_{x_j}|\, \zeta\, dx \\ &\le \int \sqrt{\langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j}\rangle} \\ &\quad \times \sqrt{\langle D^2G(\nabla u)\, \nabla \zeta, \nabla \zeta\rangle}\, |u_{x_j}|\, \zeta\, dx\end{aligned},$$

thanks to Cauchy-Schwartz inequality. By using Young's inequality in a standard fashion, from (3.5) and the previous inequality we can obtain

$$\begin{aligned}\int \langle D^2G(\nabla u)\, \nabla u_{x_j}, \nabla u_{x_j}\rangle\, \zeta^2\, dx &\le 4 \int \langle D^2G(\nabla u)\, \nabla \zeta, \nabla \zeta\rangle\, |u_{x_j}|^2\, dx \\ &\quad + 2 \int f_{x_j}\, u_{x_j}\, \zeta^2\, dx.\end{aligned}$$

We now use that

$$|z|^{p-2}\, |\xi|^2 \le \langle D^2G(z)\, \xi, \xi\rangle \le (p-1)\, (\varepsilon + |z|^2)^{\frac{p-2}{2}}\, |\xi|^2, \qquad z, \xi \in \mathbb{R}^N,$$

thus with simple manipulations we get

$$\int |\nabla u|^{p-2}\, |\nabla u_{x_j}|^2\, \zeta^2\, dx \le \frac{4\,(p-1)}{(R-r)^2} \int_{B_R} (\varepsilon + |\nabla u|^2)^{\frac{p}{2}}\, dx + 2 \int f_{x_j}\, u_{x_j}\, \zeta^2\, dx.$$

We observe that

$$|\nabla u|^{p-2}\, |\nabla u_{x_j}|^2 = \frac{4}{p^2} \left| \left(|\nabla u|^{\frac{p-2}{2}}\, \nabla u \right)_{x_j} \right|^2 = \frac{4}{p^2}\, \left|\mathcal{V}_{x_j}\right|^2, \tag{3.6}$$

thus we have

$$\int \left|\mathcal{V}_{x_j}\right|^2\, \zeta^2\, dx \le \frac{(p-1)\, p^2}{(R-r)^2} \int_{B_R} (\varepsilon + |\nabla u|^2)^{\frac{p}{2}}\, dx + \frac{p^2}{2} \int f_{x_j}\, u_{x_j}\, \zeta^2\, dx. \tag{3.7}$$

We are left with the estimate of the term containing f_{x_j}. By definition of dual norm, for[4] $(p-2)/p < s < 1$ we have

$$\int f_{x_j}\, u_{x_j}\, \zeta^2\, dx \le \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, [u_{x_j}\, \zeta^2]_{W^{1-s,p}(\mathbb{R}^N)}. \tag{3.8}$$

Observe that by assumption $0 < 1-s < 2/p$, thus by Proposition 2.4 with $\alpha = 1-s$ and $\beta = 2/p$, we have

$$[u_{x_j}\, \zeta^2]^p_{W^{1-s,p}(\mathbb{R}^N)} \le \frac{C}{\left(s - \dfrac{p-2}{p}\right)(1-s)} \left([u_{x_j}\, \zeta^2]^p_{\mathcal{N}^{\frac{2}{p},p}_\infty(\mathbb{R}^N)}\right)^{(1-s)\frac{p}{2}} \left(\|u_{x_j}\|^p_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)}, \tag{3.9}$$

In order to estimate the norm of $u_{x_j}\, \zeta^2$, we recall that

$$[u_{x_j}\, \zeta^2]^p_{\mathcal{N}^{\frac{2}{p},p}_\infty(\mathbb{R}^N)} = \sup_{|h|>0} \int_{\mathbb{R}^N} \frac{|\delta_h(u_{x_j}\, \zeta^2)|^p}{|h|^2}\, dx.$$

Then we observe that from (2.1)

$$|a-b|^p \le C\, \left||a|^{\frac{p-2}{2}}\, a - |b|^{\frac{p-2}{2}}\, b\right|^2, \qquad a, b \in \mathbb{R}.$$

Thus we obtain

$$\int_{\mathbb{R}^N} \frac{|\delta_h(u_{x_j}\, \zeta^2)|^p}{|h|^2}\, dx \le C \int_{\mathbb{R}^N} \frac{\left|\delta_h\left(|u_{x_j}|^{\frac{p-2}{2}}\, u_{x_j}\, \zeta^p\right)\right|^2}{|h|^2}\, dx \le C \int_{\mathbb{R}^N} \left|\nabla\left(|u_{x_j}|^{\frac{p-2}{2}}\, u_{x_j}\, \zeta^p\right)\right|^2\, dx,$$

where the second inequality comes from the classical characterization of $W^{1,2}$ in terms of finite differences. By recalling the properties (3.4) of ζ, with simple manipulations we thus obtain

$$[u_{x_j}\, \zeta^2]^p_{\mathcal{N}^{\frac{2}{p},p}_\infty(\mathbb{R}^N)} \le C \int \left|\nabla\left(|u_{x_j}|^{\frac{p-2}{2}}\, u_{x_j}\right)\right|^2 \zeta^2\, dx + \frac{C}{(R-r)^2} \int_{B_R} |u_{x_j}|^p\, dx,$$

for a constant $C = C(N,p) > 0$. Observe that

$$\left|\nabla\left(|u_{x_j}|^{\frac{p-2}{2}}\, u_{x_j}\right)\right|^2 = \frac{p^2}{4}\, |u_{x_j}|^{p-2}\, |\nabla u_{x_j}|^2 \le \frac{p^2}{4}\, |\nabla u|^{p-2}\, |\nabla u_{x_j}|^2 = |\mathcal{V}_{x_j}|^2,$$

thanks to (3.6). This yields

$$[u_{x_j}\, \zeta^2]^p_{\mathcal{N}^{\frac{2}{p},p}_\infty(\mathbb{R}^N)} \le C \int |\mathcal{V}_{x_j}|^2\, \zeta^2\, dx + \frac{C}{(R-r)^2} \int_{B_R} |\nabla u|^p\, dx.$$

By inserting this estimate in (3.9), from (3.8) we get

$$\left|\int f_{x_j}\, u_{x_j}\, \zeta^2\, dx\right| \le C\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)} \left(\|\mathcal{V}_{x_j}\, \zeta\|^{\frac{2}{p}}_{L^2(B_R)} + \frac{C}{(R-r)^{\frac{2}{p}}}\, \|\nabla u\|_{L^p(B_R)}\right)^{(1-s)\frac{p}{2}}$$
$$\times \left(\|\nabla u\|_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)},$$

[4] We exclude here the case $s = 1$, since this is the easy case. It would be sufficient to use Hölder's inequality with exponents p and p' to conclude.

for a constant $C = C(N, p, s) > 0$, which blows-up as $s \searrow (p-2)/p$. We can still manipulate a bit the previous estimate and obtain

$$\begin{aligned}\left|\int f_{x_j}\, u_{x_j}\, \zeta^2\, dx\right| &\le C\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)} \left(\|\mathcal{V}_{x_j}\,\zeta\|_{L^2(B_R)}^{1-s} + \frac{C}{(R-r)^{1-s}}\, \|\nabla u\|_{L^p(B_R)}^{(1-s)\frac{p}{2}}\right) \\ &\quad \times \left(\|\nabla u\|_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)} \\ &\le C\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, \|\mathcal{V}_{x_j}\,\zeta\|_{L^2(B_R)}^{1-s} \left(\|\nabla u\|_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)} \\ &\quad + \frac{C}{(R-r)^{1-s}}\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, \|\nabla u\|_{L^p(B_R)},\end{aligned}$$

for a different constant $C > 0$, still depending on N, p and s only. We now go back to (3.7) and use the previous estimate. This gives

$$\begin{aligned}\int \left|\mathcal{V}_{x_j}\right|^2 \zeta^2\, dx &\le \frac{C}{(R-r)^2} \int_{B_R} (\varepsilon + |\nabla u|^2)^{\frac{p}{2}}\, dx \\ &\quad + C\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, \|\mathcal{V}_{x_j}\,\zeta\|_{L^2(B_R)}^{1-s} \left(\|\nabla u\|_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)} \\ &\quad + \frac{C}{(R-r)^{1-s}}\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, \|\nabla u\|_{L^p(B_R)}.\end{aligned}$$

We need to absorb the higher order term containing $\mathcal{V}$ in the right-hand side. For this, we use Young's inequality with exponents

$$p', \qquad \frac{2}{1-s} \qquad \text{and} \qquad \frac{2\,p}{2-p\,(1-s)},$$

so to get

$$\begin{aligned}&\|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)} \|\mathcal{V}_{x_j}\,\zeta\|_{L^2(B_R)}^{1-s} \left(\|\nabla u\|_{L^p(B_R)}\right)^{1-\frac{p}{2}(1-s)} \\ &\qquad \le C\, \tau^{-\frac{(1-s)}{2}\,p'}\, (R-r)^{\left(s-\frac{p-2}{p}\right)p'}\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}^{p'} + \tau\, \|\mathcal{V}_{x_j}\,\zeta\|_{L^2(B_R)}^2 \\ &\qquad + \frac{C}{(R-r)^2}\, \|\nabla u\|_{L^p(B_R)}^p,\end{aligned}$$

which yields (by taking $\tau > 0$ small enough)

$$\begin{aligned}\int \left|\mathcal{V}_{x_j}\right|^2 \zeta^2\, dx &\le \frac{C}{(R-r)^2} \int_{B_R} (\varepsilon + |\nabla u|^2)^{\frac{p}{2}}\, dx + C\,(R-r)^{\left(s-\frac{p-2}{p}\right)p'}\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}^{p'} \\ &\quad + \frac{C}{(R-r)^{1-s}}\, \|f_{x_j}\|_{\mathcal{D}^{s-1,p'}(B_R)}\, \|\nabla u\|_{L^p(B_R)}.\end{aligned}$$

We now apply Young's inequality once more and Theorem 2.6, so to obtain (3.2). $\square$

4. Proof of Theorem 1.1

Let $U \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local weak solution in Ω of (1.1). We fix a ball $B_R \Subset \Omega$ and take a pair of concentric ball B and $\widetilde{B}$ such that $B_R \Subset B \Subset \widetilde{B} \Subset \Omega$. There exists $\varepsilon_0 > 0$ such that

$$\|f_\varepsilon\|_{L^{p'}(B)} + [f_\varepsilon]_{W^{s,p'}(B)} \le \|f\|_{L^{p'}(\widetilde{B})} + [f]_{W^{s,p'}(\widetilde{B})} < +\infty, \qquad \text{for every } 0 < \varepsilon < \varepsilon_0. \tag{4.1}$$

We consider for every $0<\varepsilon<\varepsilon_0$ the solution u_ε of problem (3.1) in the ball B. By (3.2) and (4.1) we know that $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is bounded in $W^{1,p}(B)$, thus by Rellich-Kondrašov Theorem we can extract a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0, such that

$$\lim_{k\to\infty}\|u_{\varepsilon_k}-u\|_{L^p(B)}=0,$$

for some $u\in W^{1,p}(B)$. Since u_{ε_k} is the unique solution of

$$\min\left\{\int_B G_{\varepsilon_k}(\nabla\varphi)\,dx-\int_B f_{\varepsilon_k}\,\varphi\,dx\,:\,\varphi-U\in W^{1,p}_0(B)\right\},$$

by a standard $\Gamma-$convergence argument we can easily show that $u=U$, i.e. the limit function coincides with our local weak solution U.

From Proposition 3.2 and boundedness of $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$, we know that $\{\mathcal{V}_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(B_r;\mathbb{R}^N)$, for every $B_r\Subset B$. Up to passing a subsequence, we can infer convergence to some vector field $\mathcal{Z}\in W^{1,2}(B_r;\mathbb{R}^N)$, weakly in $W^{1,2}(B_r;\mathbb{R}^N)$ and strongly in $L^2(B_r;\mathbb{R}^N)$. In particular, this is a Cauchy sequence in $L^2(B_r)$ and by using the elementary inequality (2.1), we obtain that $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ as well is a Cauchy sequence in $W^{1,p}(B_r)$. Thus we obtain

$$\lim_{k\to\infty}\|\nabla u_{\varepsilon_k}-\nabla U\|_{L^p(B_r)}=0.$$

We need to show that $\mathcal{Z}=|\nabla U|^{(p-2)/2}\,U$. We use the elementary inequality (2.2), this yields

$$\begin{aligned}\int_{B_r}\left||\nabla u_{\varepsilon_k}|^{\frac{p-2}{2}}\,\nabla u_{\varepsilon_k}-|\nabla U|^{\frac{p-2}{2}}\,\nabla U\right|^2\,dx&\le C\,\int_{B_r}\left(|\nabla u_{\varepsilon_k}|^{\frac{p-2}{2}}+|\nabla U|^{\frac{p-2}{2}}\right)^2\,|\nabla u^{\varepsilon_k}-\nabla U|^2\,dx\\&\le C\,\left(\int_{B_r}\left(|\nabla u_{\varepsilon_k}|^{\frac{p-2}{2}}+|\nabla U|^{\frac{p-2}{2}}\right)^{\frac{2\,p}{p-2}}\,dx\right)^{\frac{p-2}{p}}\\&\quad\times\left(\int_{B_r}|\nabla u_{\varepsilon_k}-\nabla U|^p\,dx\right)^{\frac{2}{p}}.\end{aligned}$$

By using the strong convergence of the gradients proved above, this gives that $\mathcal{Z}=|\nabla U|^{\frac{p-2}{2}}\,\nabla U$ and it belongs to $W^{1,2}(B_r;\mathbb{R}^N)$. By arbitrariness of the ball $B_r\Subset B$ in the above discussion, we can now take the ball B_R fixed at the beginning and pass to the limit in (3.3), so to obtain the desired estimate (1.4).

Finally, the fact that $\nabla U\in W^{\tau,p}_{\rm loc}(\Omega;\mathbb{R}^N)$ follows in standard way from the elementary inequality (2.1). We leave the details to the reader.

5. An example

We now show with an explicit example that the assumption on f in Theorem 1.1 is essentially sharp.

Let us take $U(x)=|x|^{-\alpha}$, which belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^N)$ if and only if

$$\alpha<\frac{N}{p}-1.$$

Then we can compute

$$|\nabla U|^{p-2}\,\nabla U\simeq|x|^{-(\alpha+1)\,(p-1)-1}\,x,\qquad \operatorname{div}\left(|\nabla U|^{p-2}\,\nabla U\right)\simeq|x|^{-(\alpha+1)\,(p-1)-1},$$

and

$$|\nabla U|^{\frac{p-2}{2}}\,\nabla U\simeq|x|^{-(\alpha+1)\,\frac{p}{2}-1}\,x.$$

We observe that

$$|\nabla U|^{\frac{p-2}{2}}\,\nabla U \in W^{1,2}_{\rm loc}(\mathbb{R}^N) \qquad \Longleftrightarrow \qquad \alpha < \frac{N-2}{p} - 1 =: \widetilde{\alpha}.$$

On the other hand the function $f(x) = |x|^{-(\alpha+1)\,(p-1)-1}$ belongs to $W^{s,p'}_{\rm loc}(\mathbb{R}^N)$ if

$$(\alpha+1)\,(p-1)+1 < \frac{N-s\,p'}{p'} \qquad \text{i. e. if} \qquad \alpha < \frac{N}{p} - \frac{s+1}{p-1} - 1 =: \alpha_s.$$

If we take $s < (p-2)/p$, we have

$$\alpha_s > \frac{N-2}{p} - 1 = \widetilde{\alpha},$$

thus for every such s, if we take $\alpha = (N-2)/p - 1$ we get

$$-\Delta_p U = f \in W^{s,p'}_{\rm loc}(\mathbb{R}^N) \qquad \text{and} \qquad |\nabla U|^{\frac{p-2}{2}}\,\nabla U \notin W^{1,2}_{\rm loc}(\mathbb{R}^N).$$

This shows that in Theorem 1.1 the differentiability exponent s of f can not go below $(p-2)/p$.

Appendix A. Tools for the proof of Theorem 2.6

A.1. **Basics of real interpolation: the $K-$method.** We first present a couple of basic facts from the theory of real interpolation, by referring the reader to [1] for more details.

Definition A.1. Let X and Y two Banach spaces over $\mathbb{R}$. We say that (X,Y) is a *compatible couple* is there exists a Hausdorff topological vector space Z such that X and Y are continuously embedded in Z.

When (X,Y) is a compatible couple, it does make sense to consider the two spaces $X\cap Y$ and $X+Y$. Thus we can give the definition of interpolation space.

Definition A.2. Let (X,Y) be a compatible couple of Banach spaces over $\mathbb{R}$. For every $u\in X+Y$ and $t>0$ we define

$$K(t,u,X,Y) = \inf_{u_1\in X,\,u_2\in Y}\Big\{\|u_1\|_X + t\,\|u_2\|_Y \,:\, u = u_1+u_2\Big\}.$$

For $0<\alpha<1$ and $1<q<\infty$, the *interpolation space* $(X,Y)_{\alpha,q}$ consists of

$$(X,Y)_{\alpha,q} = \left\{u\in X+Y \,:\, \int_0^{+\infty} t^{-\alpha\,q}\,K(t,u,X,Y)^q\,\frac{dt}{t} < +\infty\right\}.$$

This is a Banach space with the norm

$$\|u\|_{(X,Y)_{\alpha,q}} = \left(\int_0^{+\infty} t^{-\alpha\,q}\,K(t,u,X,Y)^q\,\frac{dt}{t}\right)^{\frac{1}{q}}.$$

The following result is well-known.

Lemma A.3. *Let $0<\beta<1$ and $1<q<\infty$. For every $B\subset\mathbb{R}^N$ open ball, we have*

$$W^{\beta,q}(B) = \big(L^q(B), W^{1,q}(B)\big)_{\beta,q},$$

and there exists a constant $C = C(N,\beta,q)>1$ such that

$$\frac{1}{C}\,\|u\|_{(L^q(B),W^{1,q}(B))_{\beta,q}} \le \|u\|_{W^{\beta,q}(B)} \le C\,\|u\|_{(L^q(B),W^{1,q}(B))_{\beta,q}}$$

Proof. This is classical, a proof can be found in [12, Chapters 34 & 36]. $\square$

A.2. **Weak derivatives.** For $j \in \{1,\dots,N\}$, let us consider the linear operator T_j defined by the distributional $j-$th partial derivative, i.e.

$$\langle T_j(F), \varphi\rangle = -\langle F, \varphi_{x_j}\rangle$$

for every test function φ and every distribution F. We first recall the following fact, whose proof is straightforward.

Lemma A.4. *Let $1<q<\infty$ and $j \in \{1,\dots,N\}$, the linear operators*

$$T_j : W^{1,q}(B) \to L^q(B) \qquad \text{and} \qquad T_j : L^q(B) \to \mathcal{D}^{-1,q}(B),$$

are continuous on their domains. More precisely, we have

$$\|T_j(u)\|_{L^q(B)} \le \|u\|_{W^{1,q}(B)}, \qquad \text{for every } u \in W^{1,q}(B), \tag{A.1}$$

and

$$\|T_j(u)\|_{\mathcal{D}^{-1,q}(B)} \le \|u\|_{L^q(B)}, \qquad \text{for every } u \in L^q(B). \tag{A.2}$$

We can then prove the following result.

Lemma A.5. *Let $0<\beta<1$ and $1<q<\infty$. Then the restriction of T_j to $W^{\beta,q}(B)$ is a linear continuous operator from $W^{\beta,q}(B)$ to the interpolation space*

$$\mathcal{Y}(B) := \left(\mathcal{D}^{-1,q}(B), L^q(B)\right)_{\beta,q}.$$

In other words, there exist a constant $C=C(N,\beta,q)>0$ such that

$$\int_0^{+\infty} t^{-\beta\, q}\, K(t, T_j(u), \mathcal{D}^{-1,q}(B), L^q(B))^q\, \frac{dt}{t} \le C\, \|u\|^q_{W^{\beta,q}(B)}, \qquad \text{for every } u \in W^{\beta,q}(B).$$

Proof. Let $u \in W^{\beta,q}(B)$, by Lemma A.3 there exists $u_1 \in L^q(B)$ and $u_2 \in W^{1,q}(B)$ such that

$$u = u_1 + u_2 \qquad \text{so that by linearity} \qquad T_j(u) = T_j(u_1) + T_j(u_2).$$

From the definition of the functional K, we have

$$\begin{aligned} t^{-\beta}\, K(t, T_j(u), \mathcal{D}^{-1,q}(B), L^q(B)) &\le t^{-\beta}\left(\|T_j(u_1)\|_{\mathcal{D}^{-1,q}(B)} + t\,\|T_j(u_2)\|_{L^q(B)}\right)\\ &\le t^{-\beta}\left(\|u_1\|_{L^q(B)} + t\,\|u_2\|_{W^{1,q}(B)}\right) \end{aligned}$$

where we used (A.1) and (A.2). By taking the infimum over the admissible pairs (u_1,u_2), we thus get

$$t^{-\beta}\, K(t, T_j(u), \mathcal{D}^{-1,q}(B), L^q(B)) \le t^{-\beta}\, K(t, u, L^q(B), W^{1,q}(B)).$$

By Lemma A.3 again, the right-hand side belongs to $L^q((0,+\infty);1/t)$. Thus, the same property is true for the left-hand side and this concludes the proof. $\square$

A.3. **Computation of an interpolation space.** We now compute the interpolation space occurring in Lemma A.5. In what follows we denote by X^* the topological dual of a Banach space X.

Lemma A.6 (Interpolation of dual spaces)**.** *Let $0<\beta<1$ and $1<q<\infty$, then we have the following chain of identities*

$$
\begin{aligned}
\big(\mathcal{D}^{-1,q}(B),L^q(B)\big)_{\beta,q} &= \left(\Big(\mathcal{D}_0^{1,q'}(B),L^{q'}(B)\Big)_{\beta,q'}\right)^* \\
&= \left(\Big(L^{q'}(B),\mathcal{D}_0^{1,q'}(B)\Big)_{1-\beta,q'}\right)^* \\
&= \Big(\mathcal{D}_0^{1-\beta,q'}(B)\Big)^* \\
&= \mathcal{D}^{\beta-1,q}(B),
\end{aligned}
$$

as Banach spaces.

Proof. The first identity is a consequence of the so-called *Duality Theorem* in real interpolation theory, see [1, Theorem 3.7.1]. This result requires the space

$$
L^{q'}(B)\cap\mathcal{D}_0^{1,q'}(B)=\mathcal{D}_0^{1,q'}(B),
$$

to be dense both in $L^{q'}(B)$ and $\mathcal{D}_0^{1,q'}(B)$, an hypothesis which is of course verified.

The second identity is a basic fact in real interpolation, see [1, Theorem 3.4.1]. On the other hand, the fourth identity is just the definition of dual space.

In order to conclude, we need to show the third identity, i. e.

$$
\Big(L^{q'}(B),\mathcal{D}_0^{1,q'}(B)\Big)_{1-\beta,q'}=\mathcal{D}_0^{1-\beta,q'}(B).
$$

Let us set for notational simplicity

$$
\mathcal{X}(B):=\Big(L^{q'}(B),\mathcal{D}_0^{1,q'}(B)\Big)_{1-\beta,q'}.
$$

For every $u\in L^{q'}(B)+\mathcal{D}_0^{1,q'}(B)$ and $t>0$, we also use the simplified notation

$$
K(t,u)=\inf_{u_1\in L^{q'}(B),\,u_2\in\mathcal{D}_0^{1,q'}(B)}\Big\{\|u_1\|_{L^{q'}(B)}+t\,\|u_2\|_{\mathcal{D}_0^{1,q'}(B)}\;:\;u=u_1+u_2\Big\}.
$$

For every $u\in\mathcal{X}\setminus\{0\}$ and $h\in\mathbb{R}^N\setminus\{0\}$, there exist $u_1\in L^{q'}(B)$ and $u_2\in\mathcal{D}_0^{1,q'}(B)$ such that

$$
u=u_1+u_2\qquad\text{and}\qquad\|u_1\|_{L^{q'}(B)}+|h|\,\|u_2\|_{\mathcal{D}_0^{1,q'}(B)}\le 2\,K(|h|,u). \tag{A.3}
$$

Both u_1 and u_2 are extended to $\mathbb{R}^N\setminus\overline{B}$ by 0. Thus for $h\neq 0$ we get[5]

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^{q'}}{|h|^{N+(1-\beta)\,q'}}\,dx &\le C\int_{\mathbb{R}^N}\frac{|u_1(x+h)-u_1(x)|^{q'}}{|h|^{N+(1-\beta)\,q'}}\,dx\\ &+C\int_{\mathbb{R}^N}\frac{|u_2(x+h)-u_2(x)|^{q'}}{|h|^{N+(1-\beta)\,q'}}\,dx\\ &\le C\,|h|^{-N-(1-\beta)\,q'}\,\|u_1\|^{q'}_{L^{q'}(B)}\\ &+C\,|h|^{-N+\beta\,q'}\,\|\nabla u_2\|^{q'}_{L^{q'}(B)}\\ &\le C\,|h|^{-N-(1-\beta)\,q'}\,\Big(\|u_1\|_{L^{q'}(B)}+|h|\,\|u_2\|_{\mathcal{D}^{1,q'}_0(B)}\Big)^{q'}.\end{aligned}$$

By using (A.3), we obtain

$$\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^{q'}}{|h|^{N+(1-\beta)\,q'}}\,dx\le C\,|h|^{-N-(1-\beta)\,q'}\,K(|h|,u)^{q'}.$$

We now integrate in h over $\mathbb{R}^N$. This yields

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x+h)-u(x)|^{q'}}{|h|^{N+(1-\beta)\,q'}}\,dx\,dh&\le C\int_{\mathbb{R}^N}|h|^{-N-(1-\beta)\,q'}\,K(|h|,u)^{q'}\,dh\\ &=C\,N\,\omega_N\int_0^{+\infty}t^{(\beta-1)\,q'}\,K(t,u)^{q'}\,\frac{dt}{t}=C\,\|u\|^{q'}_{\mathcal{X}(B)},\end{aligned}$$

which shows that we have the continuous embedding

$$\mathcal{X}(B)\hookrightarrow\mathcal{D}^{1-\beta,q'}_0(B).$$

We now need to show the converse embedding. Let us indicate by $R>0$ the radius of B, we further assume for simplicity that B is centered at the origin. Let $u\in\mathcal{D}^{1-\beta,q'}_0(B)$, then in particular $u\in L^{q'}(B)$ (see Remark 2.2) and thus from the writing

$$u=u+0,$$

by definition of K we obtain

$$\begin{aligned}\int_{R/2}^{+\infty}t^{(\beta-1)\,q'}\,K(t,u)^{q'}\,\frac{dt}{t}&\le\left(\int_{R/2}^{+\infty}t^{(\beta-1)\,q'-1}\,dt\right)\|u\|^{q'}_{L^{q'}(B)}\\ &=\frac{R^{(\beta-1)\,q'}}{2^{(\beta-1)\,q'}\,(1-\beta)\,q'}\,\|u\|^{q'}_{L^{q'}(B)}\le C\,\|u\|^{q'}_{\mathcal{D}^{1-\beta,q'}_0(B)},\end{aligned}\tag{A.4}$$

for some $C=C(N,\beta,q)>0$. In the last passage we used Poincaré inequality for $\mathcal{D}^{1-\beta,q'}_0(B)$.

[5]In the second inequality, we use the classical fact

$$\int_{\mathbb{R}^N}\frac{|u_2(x+h)-u_2(x)|^{q'}}{|h|^{q'}}\,dx\le C\int_{\mathbb{R}^N}|\nabla u_2|^{q'}\,dx.$$

We need to deal with $0 < t < R/2$. For this, we take a positive function $\psi \in C_0^\infty$ with support contained in $\{x \in \mathbb{R}^N : |x| < 1\}$ and such that $\int_{\mathbb{R}^N} \psi\, dx = 1$. Then we define

$$\psi_t(x) = \frac{1}{t^N}\, \psi\left(\frac{x}{t}\right), \qquad x \in \mathbb{R}^N,\ t > 0,$$

and

$$v_t(x) = u\left(\frac{R}{R-t}\, x\right), \qquad x \in \mathbb{R}^N,\ 0 < t < \frac{R}{2},$$

where we intend that u is extended by 0 to the whole $\mathbb{R}^N$. In particular, v_t vanishes outside B_{R-t}. Finally, for $0 < t < R/2$ we set

$$u_t(x) = v_t * \psi_t(x) = \int_{B_{R-t}} u\left(\frac{R}{R-t}\, y\right) \frac{1}{t^N}\, \psi\left(\frac{x-y}{t}\right) dy.$$

By construction, for every $0 < t < R/2$ the function v_t is such that

$$\int_{\mathbb{R}^N} |\nabla u_t|^{q'}\, dx < +\infty \qquad \text{and} \qquad u_t \equiv 0 \ \text{ in } \mathbb{R}^N \setminus B,$$

thus $u_t \in \mathcal{D}_0^{1,q'}(B)$. We observe that by Jensen inequality

$$\|u - u_t\|_{L^{q'}(B)}^{q'} \le \int_B \int_{B_{R-t}} \left| u(x) - u\left(\frac{R}{R-t}\, y\right)\right|^{q'} \frac{1}{t^N}\, \psi\left(\frac{x-y}{t}\right) dy\, dx.$$

Thus by using a change of variable and Fubini Theorem we get

$$\begin{aligned}
&\int_0^{R/2} t^{(\beta-1)\, q'}\, \|u - u_t\|_{L^{q'}(B)}^{q'}\, \frac{dt}{t} \\
&\quad\le \int_0^{R/2} \int_B \int_{B_{R-t}} t^{(\beta-1)\, q'} \left| u(x) - u\left(\frac{R}{R-t}\, y\right)\right|^{q'} \frac{1}{t^N}\, \psi\left(\frac{x-y}{t}\right) dy\, dx\, \frac{dt}{t} \\
&\quad= \int_0^{R/2} \int_B \int_B \left(\frac{R-t}{R}\right)^N t^{(\beta-1)\, q'}\, |u(x) - u(z)|^{q'}\, \frac{1}{t^N}\, \psi\left(\frac{x}{t} - \frac{R-t}{R\, t}\, z\right) dz\, dx\, \frac{dt}{t} \\
&\quad\le \int_B \int_B |u(x) - u(z)|^{q'} \left(\int_0^{R/2} t^{(\beta-1)\, q' - N}\, \psi\left(\frac{x-z}{t} + \frac{z}{R}\right) \frac{dt}{t}\right) dz\, dx.
\end{aligned}$$

We now observe that

$$\left|\frac{x-z}{t} + \frac{z}{R}\right| > 1 \qquad \Longrightarrow \qquad \psi\left(\frac{x-z}{t} + \frac{z}{R}\right) = 0,$$

thus in particular

$$\left|\frac{x-z}{t}\right| > 1 + \left|\frac{z}{R}\right| \qquad \Longrightarrow \qquad \psi\left(\frac{x-z}{t} + \frac{z}{R}\right) = 0.$$

Finally, for $x, z \in B$ we get

$$\begin{aligned}\int_0^{R/2} t^{(\beta-1)\,q'-N}\,\psi\left(\frac{x-z}{t}+\frac{z}{R}\right)\frac{dt}{t} &\le \int_0^{+\infty} t^{(\beta-1)\,q'-N}\,\psi\left(\frac{x-z}{t}+\frac{z}{R}\right)\frac{dt}{t}\\ &= \int_{\frac{|x-z|}{1+\frac{|z|}{R}}}^{+\infty} t^{(\beta-1)\,q'-N}\,\psi\left(\frac{x-z}{t}+\frac{z}{R}\right)\frac{dt}{t}\\ &\le \int_{\frac{|x-z|}{2}}^{+\infty} t^{(\beta-1)\,q'-N}\,\psi\left(\frac{x-z}{t}+\frac{z}{R}\right)\frac{dt}{t}\\ &\le C\,\|\psi\|_{L^\infty(\mathbb{R}^N)}\,|x-z|^{-N-(1-\beta)\,q'}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_0^{R/2} t^{(\beta-1)\,q'}\,\|u-u_t\|^{q'}_{L^{q'}(B)}\,\frac{dt}{t} &\le C\,\|\psi\|_{L^\infty(\mathbb{R}^N)}\int_B\int_B \frac{|u(x)-u(z)|^{q'}}{|x-z|^{N+(1-\beta)\,q'}}\,dx\,dz\\ &\le C'\,\|u\|^{q'}_{\mathcal{D}_0^{1-\beta,q'}(B)}.\end{aligned}\tag{A.5}$$

In order to finish the proof, we just need to show that

$$\int_0^{R/2} t^{(\beta-1)\,q'}\,t^{q'}\,\|u_t\|^{q'}_{\mathcal{D}_0^{1,q'}(B)}\,\frac{dt}{t} \le C\,\|u\|^{q'}_{\mathcal{D}_0^{1-\beta,q'}(B)}.\tag{A.6}$$

We first observe that (recall that $u \equiv 0$ outside B)

$$\nabla u_t(x) = \int_{B_{R-t}} u\left(\frac{R}{R-t}\,y\right)\frac{1}{t^{N+1}}\,\nabla\psi\left(\frac{x-y}{t}\right)dy = \int_{\mathbb{R}^N} u\left(\frac{R}{R-t}\,y\right)\frac{1}{t^{N+1}}\,\nabla\psi\left(\frac{x-y}{t}\right)dy,$$

and by the Divergence Theorem

$$\int_{\mathbb{R}^N}\frac{1}{t^{N+1}}\,\nabla\psi\left(\frac{x-y}{t}\right)dy = 0.$$

Thus we obtain as well

$$-\nabla u_t(x) = \int_{\mathbb{R}^N}\left[u\left(\frac{R}{R-t}\,x\right)-u\left(\frac{R}{R-t}\,y\right)\right]\frac{1}{t^{N+1}}\,\nabla\psi\left(\frac{x-y}{t}\right)dy,$$

and by Hölder inequality

$$\begin{aligned}\|u_t\|^{q'}_{\mathcal{D}_0^{1,q'}(B)} &= \int_{\mathbb{R}^N}|\nabla u_t|^{q'}\,dx\\ &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left|u\left(\frac{R}{R-t}\,x\right)-u\left(\frac{R}{R-t}\,y\right)\right|^{q'}\frac{1}{t^{N+1}}\left|\nabla\psi\left(\frac{x-y}{t}\right)\right|dy\right)\\ &\times\left(\int_{\mathbb{R}^N}\frac{1}{t^{N+1}}\left|\nabla\psi\left(\frac{x-y}{t}\right)\right|dy\right)^{q'-1}dx\\ &= \frac{\|\nabla\psi\|^{q'-1}_{L^1(\mathbb{R})}}{t^{q'-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|u\left(\frac{R}{R-t}\,x\right)-u\left(\frac{R}{R-t}\,y\right)\right|^{q'}\frac{1}{t^{N+1}}\left|\nabla\psi\left(\frac{x-y}{t}\right)\right|dy\,dx\\ &\le \frac{\|\nabla\psi\|^{q'-1}_{L^1(\mathbb{R})}}{t^{q'-1}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u\,(z)-u\,(w)|^{q'}\,\frac{1}{t^{N+1}}\left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right|dz\,dw.\end{aligned}$$

This yields

$$
\begin{aligned}
&\int_0^{R/2} t^{(\beta-1)\,q'}\, t^{q'}\, \|u_t\|^{q'}_{\mathcal{D}^{1,q'}_0(B)}\, \frac{dt}{t}\\
&\le C \int_0^{R/2} t^{(\beta-1)\,q'} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u\,(z)-u\,(w)|^{q'}\, \frac{1}{t^{N+1}} \left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right| dz\,dw\,dt\\
&= C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(z)-u(w)|^{q'} \left(\int_0^{R/2} t^{(\beta-1)\,q'}\, \frac{1}{t^N} \left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right| \frac{dt}{t}\right) dz\,dw\\
&\le C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(z)-u(w)|^{q'} \left(\int_0^{+\infty} t^{(\beta-1)\,q'}\, \frac{1}{t^N} \left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right| \frac{dt}{t}\right) dz\,dw.
\end{aligned}
\tag{A.7}
$$

We now observe that

$$
\frac{R-t}{R}\,\frac{|z-w|}{t} > 1 \qquad \Longrightarrow \qquad \nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right) = 0,
$$

thus in particular for $0 < t < R/2$ we have

$$
\frac{1}{2}\,\frac{|z-w|}{t} > 1 \qquad \Longrightarrow \qquad \nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right) = 0.
$$

This implies that for $z, w \in \mathbb{R}^N$ we have

$$
\begin{aligned}
\int_0^{\infty} t^{(\beta-1)\,q'}\, \frac{1}{t^N} \left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right| \frac{dt}{t} &= \int_{\frac{|z-w|}{2}}^{+\infty} t^{(\beta-1)\,q'}\, \frac{1}{t^N} \left|\nabla\psi\left(\frac{R-t}{R\,t}\,(z-w)\right)\right| \frac{dt}{t}\\
&\le C\, \|\nabla\psi\|_{L^\infty(\mathbb{R}^N)}\, |z-w|^{-N-(1-\beta)\,q'}.
\end{aligned}
$$

Then from (A.7) we obtain (A.6). We can now conclude the proof of the embedding

$$
\mathcal{D}^{1-\beta,q'}_0(B) \hookrightarrow \mathcal{X}(B).
$$

From (A.4), (A.5) and (A.6) we get

$$
\begin{aligned}
\int_0^{+\infty} t^{(\beta-1)\,q'}\, K(t,u)^{q'}\, \frac{dt}{t} &= \int_0^{R/2} t^{(\beta-1)\,q'}\, K(t,u)^{q'}\, \frac{dt}{t} + \int_{R/2}^{+\infty} t^{(\beta-1)\,q'}\, K(t,u)^{q'}\, \frac{dt}{t}\\
&\le C \int_0^{R/2} t^{(\beta-1)\,q'}\, \|u-u_t\|^{q'}\, \frac{dt}{t}\\
&\quad + C \int_0^{R/2} t^{(\beta-1)\,q'}\, t^{q'}\, \|u_t\|^{q'}_{\mathcal{D}^{1,q'}_0(B)}\, \frac{dt}{t}\\
&\quad + \int_{R/2}^{+\infty} t^{(\beta-1)\,q'}\, K(t,u)^{q'}\, \frac{dt}{t} \le C\, \|u\|^{q'}_{\mathcal{D}^{1-\beta,q'}_0(B)},
\end{aligned}
$$

for some $C = C(N,\beta,q) > 0$. This concludes the proof. □

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